EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

Sol. Given that $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x+3)(x-3)}{(x-3)} = \lim\limits_{x \to 3} x + 3$

Taking limit, we have

$$3 + 3 = 6$$

Hence, the answer is 6.

Q2. Evaluate $\lim\limits_{x \to \frac{1}{2}} \dfrac{4x^2 - 1}{2x - 1}$

Sol. Given that: $\lim\limits_{x \to \frac{1}{2}} \dfrac{4x^2 - 1}{2x - 1}$

$$= \lim_{x \to \frac{1}{2}} \frac{(2x)^2 - (1)^2}{2x - 1} = \lim_{x \to \frac{1}{2}} \frac{(2x+1)(2x-1)}{2x - 1}$$

$$= \lim_{x \to \frac{1}{2}} (2x + 1)$$

Taking limit, we have

$$= 2 \times \frac{1}{2} + 1 = 1 + 1 = 2$$

Hence, the answer is 2.

Q3. Evaluate: $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

Sol. Given that $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h[\sqrt{x+h} + \sqrt{x}]} \times \sqrt{x+h} + \sqrt{x}$$

[Rationalizing the denominator]

$$= \lim_{h \to 0} \frac{x + h - x}{h[\sqrt{x+h} + \sqrt{x}]}$$

$$= \lim_{h \to 0} \frac{h}{h[\sqrt{x+h} + \sqrt{x}]} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Taking the limits, we have

$$\frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Hence, the answer is $\frac{1}{2\sqrt{x}}$.

Q4. Evaluate: $\lim_{x \to 0} \frac{(x+2)^{1/3} - 2^{1/3}}{x}$

Sol. Given that $\lim_{x \to 0} \frac{(x+2)^{1/3} - 2^{1/3}}{x}$

Put $x + 2 = y \quad \Rightarrow \quad x = y - 2$

$$= \lim_{y-2 \to 0} \frac{y^{1/3} - 2^{1/3}}{y - 2} = \lim_{y \to 2} \frac{y^{1/3} - 2^{1/3}}{y - 2}$$

$$= \frac{1}{3} \cdot (2)^{\frac{1}{3}-1} = \frac{1}{3} \cdot 2^{-2/3} \qquad \left[\text{using } \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$$

Hence the answer is $\frac{1}{3}(2)^{-2/3}$

Q5. Evaluate: $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

Sol. Given that: $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

Dividing the numerator and denominator by x, we get

$$= \lim_{x \to 0} \frac{\dfrac{(1+x)^6 - 1}{x}}{\dfrac{(1+x)^2 - 1}{x}}$$

Putting $1 + x = y \quad \Rightarrow \quad x = y - 1$

$$= \lim_{\substack{y-1 \to 0 \\ \therefore y \to 1}} \frac{\dfrac{y^6 - (1)^6}{y - 1}}{\dfrac{y^2 - (1)^2}{y - 1}} = \frac{\lim_{y \to 1} \dfrac{y^6 - (1)^6}{y - 1}}{\lim_{y \to 1} \dfrac{y^2 - (1)^2}{y - 1}} \qquad \left[\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{6 \cdot (1)^{6-1}}{2 \cdot (1)^{2-1}} = \frac{6}{2} = 3 \qquad \left[\text{using } \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$$

Hence, the required answer is 3.

Q6. Evaluate: $\lim\limits_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a}$

Sol. Given that: $\lim\limits_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{(2+x) - (a+2)}$

$$= \lim_{2+x \to a+2} \frac{(2+x)^{5/2} - (a+2)^{5/2}}{(2+x) - (a+2)}$$

$$= \frac{5}{2}(a+2)^{5/2-1} = \frac{5}{2}(a+2)^{3/2} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$$

Hence, the required answer is $\frac{5}{2}(a+2)^{3/2}$.

Q7. Evaluate: $\lim\limits_{x \to 1} \dfrac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$

Sol. Given that $\lim\limits_{x \to 1} \dfrac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$

$$= \lim_{x \to 1} \frac{\sqrt{x}\,[(x)^{7/2} - 1]}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{\sqrt{x}\,\dfrac{[x^{7/2} - (1)^{7/2}]}{x - 1}}{\dfrac{(x)^{1/2} - (1)^{1/2}}{x - 1}}$$

[Dividing the numerator and denominator of x –1]

$$= \lim_{x \to 1} \frac{\dfrac{(x)^{7/2} - (1)^{7/2}}{x - 1}}{\dfrac{(x)^{1/2} - (1)^{1/2}}{x - 1}} \times \lim_{x \to 1} \sqrt{x}$$

$$\left[\because \lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)\right]$$

$$= \frac{\dfrac{7}{2}(1)^{7/2-1}}{\dfrac{1}{2}(1)^{1/2-1}} \times \sqrt{1} = \frac{7/2}{1/2} = 7$$

Hence the required answer is 7.

Q8. Evaluate: $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$

Sol. Given that $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$

Rationalizing the denominator, we get

$$= \lim_{x \to 2} \frac{(x-2)(x+2)[\sqrt{3x-2} + \sqrt{x+2}]}{[\sqrt{3x-2} - \sqrt{x+2}][\sqrt{3x-2} + \sqrt{x+2}]}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)[\sqrt{3x-2}+\sqrt{x+2}]}{3x-2-x-2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)[\sqrt{(3x-2)}+\sqrt{x+2}]}{2x-4}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)[\sqrt{(3x-2)}+\sqrt{x+2}]}{2(x-2)}$$

$$= \lim_{x\to 2} \frac{(x+2)[\sqrt{3x-2}+\sqrt{x+2}]}{2}$$

Taking limits, we have

$$= \frac{(2+2)[\sqrt{6-2}+\sqrt{2+2}]}{2} = \frac{4[2+2]}{2} = \frac{4\times 4}{2} = 8$$

Hence, the required answer is 8.

Q9. Evaluate: $\lim\limits_{x\to\sqrt{2}} \dfrac{x^4-4}{x^2+3\sqrt{2}x-8}$

Sol. Given that $\lim\limits_{x\to\sqrt{2}} \dfrac{x^4-4}{x^2+3\sqrt{2}x-8}$

$$= \lim_{x\to\sqrt{2}} \frac{(x^2-2)(x^2+2)}{x^2+4\sqrt{2}x-\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})(x-\sqrt{2})(x^2+2)}{x(x+4\sqrt{2})-\sqrt{2}(x+4\sqrt{2})}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})(x-\sqrt{2})(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})} = \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})(x^2+2)}{x+4\sqrt{2}}$$

Taking limits we have

$$= \frac{(\sqrt{2}+\sqrt{2})(2+2)}{\sqrt{2}+4\sqrt{2}} = \frac{2\sqrt{2}\times 4}{5\sqrt{2}} = \frac{8}{5}$$

Hence, the required answer is $\dfrac{8}{5}$.

Q10. Evaluate $\lim\limits_{x\to 1} \dfrac{x^7-2x^5+1}{x^3-3x^2+2}$

Sol. Given that

$$= \lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1} \frac{x^7-x^5-x^5+1}{x^3-x^2-2x^2+2} = \lim_{x\to 1} \frac{x^5(x^2-1)-1(x^5-1)}{x^2(x-1)-2(x^2-1)}$$

Dividing the numerator and denominator by $(x-1)$ we get

$$= \lim_{x \to 1} \frac{x^5\left(\frac{x^2-1}{x-1}\right) - 1\left(\frac{x^5-1}{x-1}\right)}{x^2\left(\frac{x-1}{x-1}\right) - 2\left(\frac{x^2-1}{x-1}\right)}$$

$$= \frac{\lim\limits_{x \to 1} x^5(x+1) - \lim\limits_{x \to 1}\left(\frac{x^5-(1)^5}{x-1}\right)}{\lim\limits_{x \to 1} x^2 - 2\lim\limits_{x \to 1}(x+1)}$$

$$= \frac{1(2) - 5 \cdot (1)^{5-1}}{1-2(2)} = \frac{2-5}{1-4} = \frac{-3}{-3} = 1$$

Hence, the required answer is 1.

Q11. Evaluate: $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$

Sol. Given that $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$

$$= \lim_{x \to 0} \frac{\left[\sqrt{1+x^3} - \sqrt{1-x^3}\right]\left[\sqrt{1+x^3} + \sqrt{1-x^3}\right]}{x^2\left[\sqrt{1+x^3} + \sqrt{1-x^3}\right]}$$

$$= \lim_{x \to 0} \frac{(1+x^3) - (1-x^3)}{x^2\left[\sqrt{1+x^3} + \sqrt{1-x^3}\right]}$$

$$= \lim_{x \to 0} \frac{1+x^3 - 1 + x^3}{x^2\left[\sqrt{1+x^3} + \sqrt{1-x^3}\right]}$$

$$= \lim_{x \to 0} \frac{2x^3}{x^2\left[\sqrt{1+x^3} + \sqrt{1-x^3}\right]} = \lim_{x \to 0} \frac{2x}{\sqrt{1+x^3} + \sqrt{1-x^3}} = 0$$

Hence, the required answer is 0.

Q12. Evaluate: $\lim\limits_{x \to 3} \dfrac{x^3 + 27}{x^5 + 243}$

Sol. Given that $\lim\limits_{x \to 3} \dfrac{x^3 + 27}{x^5 + 243}$

$$= \lim_{x \to 3} \frac{\dfrac{x^3 + (3)^3}{x-3}}{\dfrac{x^5 + (3)^5}{x-3}} \quad \text{[Dividing the Nr and Den. by } x - 3\text{]}$$

$$= \frac{\lim_{x \to 3}\left(\frac{x^3 - (-3)^3}{x+3}\right)}{\lim_{x \to 3}\left(\frac{x^5 - (-3)^5}{x+3}\right)} \qquad \left[\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{3(-3)^{3-1}}{5(-3)^{5-1}} = \frac{3 \times (-3)^2}{5 \times (-3)^4} = \frac{1}{5 \times 3} = \frac{1}{15}$$

Hence, the required answer is $\frac{1}{15}$.

Q13. Evaluate: $\lim_{x \to \frac{1}{2}}\left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1}\right)$

Sol. Given that $\lim_{x \to \frac{1}{2}}\left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1}\right)$

$$= \lim_{x \to \frac{1}{2}}\left[\frac{(8x-3)(2x+1) - (4x^2+1)}{(4x^2-1)}\right]$$

$$= \lim_{x \to \frac{1}{2}}\left[\frac{16x^2 - 6x + 8x - 3 - 4x^2 - 1}{4x^2 - 1}\right]$$

$$= \lim_{x \to \frac{1}{2}}\left[\frac{12x^2 + 2x - 4}{4x^2 - 1}\right] = \lim_{x \to \frac{1}{2}} \frac{2(6x^2 + x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to \frac{1}{2}} \frac{2[6x^2 + 4x - 3x - 2]}{(2x+1)(2x-1)} = \lim_{x \to \frac{1}{2}} \frac{2[2x(3x+2) - 1(3x+2)]}{(2x+1)(2x-1)}$$

$$= \lim_{x \to \frac{1}{2}} \frac{2(3x+2)(2x-1)}{(2x+1)(2x-1)} = \lim_{x \to \frac{1}{2}} \frac{2(3x+2)}{(2x+1)}$$

Taking limit, we have

$$= \frac{2\left(3 \times \frac{1}{2} + 2\right)}{2 \times \frac{1}{2} + 1} = \frac{2\left(\frac{7}{2}\right)}{2} = \frac{7}{2}$$

Hence, the required answer is $\frac{7}{2}$.

Q14. Find 'n' if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80,\ x \in \mathrm{N}$

Sol. Given that $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$

$= n \cdot (2)^{n-1} = 80$ $\left[\because \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$

$= n \times 2^{n-1} = 5 \times (2)^{5-1}$

$\therefore \quad n = 5$

Hence, the required answer is $n = 5$.

Q15. Evaluate: $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

Sol. Given that $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

$$= \lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \times 3x}{\frac{\sin 7x}{7x} \times 7x} = \frac{\lim_{3x \to 0}\left(\frac{\sin 3x}{3x}\right)}{\lim_{7x \to 0}\left(\frac{\sin 7x}{7x}\right)} \times \frac{3}{7}$$

$$= \frac{1}{1} \times \frac{3}{7} = \frac{3}{7} \qquad \left[\because \quad \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Hence, the required answer is $\frac{3}{7}$.

Q16. Evaluate: $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$

Sol. Given that $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$

$$= \lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 2(2x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 2x}{4 \sin^2 2x \cdot \cos^2 2x} \qquad [\sin 2x = 2 \sin x \cos x]$$

$$= \frac{1}{4 \cos^2 2x}$$

Taking limit we have

$$= \frac{1}{4 \cdot \cos^2 0} = \frac{1}{4}$$

Hence, the required answer is $\frac{1}{4}$.

Q17. Evaluate: $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

Sol. Given that $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

$$= \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} \qquad [\cos 2x = 1 - 2 \sin^2 x]$$

$$= \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)^2 = 2\times 1 = 2 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Hence, the required answer is 2.

Q18. Evaluate: $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$

Sol. Given that $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$

$$= \lim_{x\to 0} \frac{2\sin x - 2\sin x\cos x}{x^3} = \lim_{x\to 0} \frac{2\sin x(1-\cos x)}{x^3}$$

$$= \lim_{x\to 0} \frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right) = \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)\left(\frac{2\sin^2 x/2}{x^2}\right)$$

$$= \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)\left(2\,\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}\times\frac{1}{4}\right)$$

$$= \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)2\left[\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right]\cdot\frac{1}{4}$$

$$= \lim_{x\to 0} \frac{4}{4}\left(\frac{\sin x}{x}\right)\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= 1\cdot 1\cdot (1)^2 = 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Hence, the required answer is 1.

Q19. Evaluate: $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$

Sol. Given that $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$

$$= \lim_{x\to 0} \left(\frac{2\sin^2\frac{m}{2}x}{2\sin^2\frac{n}{2}x}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin\frac{m}{2}x}{\sin\frac{n}{2}x}\right)^2$$

$$= \frac{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{m}{2}x}{\frac{m}{2}x}\times\frac{m}{2}x\right)^2}{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{n}{2}x}{\sin\frac{n}{2}x}\times\frac{n}{2}x\right)^2} = \frac{1\cdot\frac{m^2}{4}x^2}{1\cdot\frac{n^2}{4}x^2} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Hence, the required answer is $\dfrac{m^2}{n^2}$.

Q20. Evaluate: $\lim\limits_{x\to\frac{\pi}{3}} \dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

Sol. Given that $\lim\limits_{x\to\frac{\pi}{3}} \dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} \qquad [\because\ 1-\cos\theta = 2\sin^2\ \theta/2]$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi-3x}{3}\right)} = \lim_{\substack{x\to\frac{\pi}{3}\\ \therefore \pi-3x\to 0}} \frac{3\cdot\sin(\pi-3x)}{\pi-3x}$$

$$= 3 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Hence, the required answer is 3.

Q21. Evaluate: $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

Sol. Given that $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x\right)}{x-\frac{\pi}{4}}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\left(\cos\frac{\pi}{4}\sin x-\sin\frac{\pi}{4}\cos x\right)}{x-\frac{\pi}{4}}$$

$$= \lim_{\substack{x \to \frac{\pi}{4} \\ \therefore x - \frac{\pi}{4} \to 0}} \frac{\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$$

$$\sqrt{2} \cdot 1 = \sqrt{2}$$

Hence, the required answer is $\sqrt{2}$.

Q22. $\lim_{x \to \frac{\pi}{6}} \dfrac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$

Sol. Given that $\lim_{x \to \frac{\pi}{6}} \dfrac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$

$$= \lim_{x \to \frac{\pi}{6}} \frac{2\left[\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x\right]}{x - \frac{\pi}{6}}$$

$$= \lim_{x \to \frac{\pi}{6}} = \frac{2\left[\cos \frac{\pi}{6} \sin x - \sin \frac{\pi}{6} \cos x\right]}{x - \frac{\pi}{6}}$$

$$= \lim_{\substack{x \to \frac{\pi}{6} \\ \therefore x - \frac{\pi}{6} \to 0}} \frac{2 \sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2 \cdot 1 = 2$$

Hence, the required answer is 2.

Q23. Evaluate: $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x}$

Sol. Given that: $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x}$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 2x + 3x}{2x}\right) \times 2x}{\left(\frac{2x + \tan 3x}{3x}\right) \times 3x} = \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{2x} + \frac{3x}{2x}\right) \times 2x}{\left(\frac{2x}{3x} + \frac{\tan 3x}{3x}\right) \times 3x}$$

$$= \frac{\left(\lim_{2x \to 0} \frac{\sin 2x}{2x} + \frac{3}{2}\right)}{\left[\frac{2}{3} + \lim_{3x \to 0} \frac{\tan 3x}{3x}\right]} \times \frac{2}{3} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \left(\frac{1+\frac{3}{2}}{\frac{2}{3}+1}\right) \times \frac{2}{3} \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$= \frac{5/2}{5/3} \times \frac{2}{3} = \frac{3}{2} \times \frac{2}{3} = 1$$

Hence, the required answer is 1.

Q24. Evaluate: $\lim_{x\to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

Sol. Given that: $\lim_{x\to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

$$= \lim_{x\to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x\to a} \frac{(\sin x - \sin a)(\sqrt{x} + \sqrt{a})}{x - a}$$

$$= \lim_{x\to a} \frac{\left(2\cos\frac{x+a}{2} \cdot \sin\frac{x-a}{2}\right)(\sqrt{x} + \sqrt{a})}{x - a}$$

$$= \lim_{\substack{x\to a \\ \therefore \frac{x-a}{2}\to 0}} \left(2\cos\frac{x+a}{2} \cdot \frac{\sin\frac{x-a}{2}}{2 \times \frac{x-a}{2}}\right)(\sqrt{x} + \sqrt{a})$$

$$= \lim_{x\to a} \cos\left(\frac{x+a}{2}\right)(\sqrt{x} + \sqrt{a}) \qquad \left[\because \lim_{\frac{x-a}{2}\to 0} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = 1\right]$$

Taking limit we have

$$= \cos\left(\frac{a+a}{2}\right)(\sqrt{a} + \sqrt{a}) = \cos a \times 2\sqrt{a} = 2\sqrt{a} \cdot \cos a$$

Hence, the required answer is $2\sqrt{a}\cos a$.

Q25. Evaluate: $\lim_{x\to\frac{\pi}{6}} \dfrac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

Sol. Given that $\lim_{x\to\frac{\pi}{6}} \dfrac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

$$= \lim_{x\to\frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2} = \lim_{x\to\frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{(\operatorname{cosec} x + 2)(\operatorname{cosec} x - 2)}{(\operatorname{cosec} x - 2)} = \lim_{x \to \frac{\pi}{6}} (\operatorname{cosec} x + 2)$$

Taking limit we have

$$= \operatorname{cosec} \frac{\pi}{6} + 2 = 2 + 2 = 4$$

Hence, the required answer is 4.

Q26. Evaluate: $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

Sol. Given that $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sqrt{2} + \sqrt{1 + \cos x}}$$

$$= \lim_{x \to 0} \frac{2 - (1 + \cos x)}{\sin^2 x \left[\sqrt{2} + \sqrt{1 + \cos x}\right]}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x \left[\sqrt{2} + \sqrt{1 + \cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 x/2}{(2 \sin x/2 \cos x/2)^2} \times \frac{1}{\left[\sqrt{2} + \sqrt{1 + \cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 x/2}{4 \sin^2 x/2 \cos^2 x/2} \times \frac{1}{\left[\sqrt{2} + \sqrt{1 + \cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2}{4 \cos^2 \frac{x}{2}} \times \frac{1}{\left[\sqrt{2} + \sqrt{1 + \cos x}\right]}$$

Taking limit, we get

$$= \frac{2}{4 \cos^2 0} \times \frac{1}{(\sqrt{2} + \sqrt{2})} = \frac{1}{2} \times \frac{1}{2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$

Hence, the required answer is $\frac{1}{4\sqrt{2}}$.

Q27. Evaluate: $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$

Sol. Given that: $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$

$$= \lim_{x \to 0} \frac{\sin x}{x} - \frac{2 \sin 3x}{x} + \frac{\sin 5x}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - \lim_{3x \to 0} 2\left(\frac{\sin 3x}{3x}\right) \times 3 + \lim_{5x \to 0}\left(\frac{\sin 5x}{5x}\right) \times 5$$

$= 1 - 6 + 5 = 0$

Hence, the required answer is 0.

Q28. If $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Sol. Given that $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$

$$\Rightarrow \quad 4(1)^{4-1} = \lim_{x \to k} \frac{(x-k)(x^2 + k^2 + kx)}{(x-k)(x+k)}$$

$$\Rightarrow \quad 4 = \lim_{x \to k} \frac{x^2 + k^2 + kx}{x + k} \quad \Rightarrow \quad 4 = \frac{k^2 + k^2 + k^2}{2k}$$

$$\Rightarrow \quad 4 = \frac{3k^2}{2k} \quad \Rightarrow \quad 4 = \frac{3}{2}k \quad \Rightarrow \quad k = \frac{8}{3}$$

Hence, the required value of k is $\dfrac{8}{3}$.

Differentiate each of the following functions from Exercise 29 to 42.

Q29. $\dfrac{x^4 + x^3 + x^2 + 1}{x}$

Sol. $\dfrac{d}{dx}\left(\dfrac{x^4 + x^3 + x^2 + 1}{x}\right) = \dfrac{d}{dx}\left(x^3 + x^2 + x + \dfrac{1}{x}\right)$

$$= 3x^2 + 2x + 1 - \frac{1}{x^2}$$

Hence, the required answer is $3x^2 + 2x + 1 - \dfrac{1}{x^2}$.

Q30. $\left(x + \dfrac{1}{x}\right)^3$

Sol. $\dfrac{d}{dx}\left(x + \dfrac{1}{x}\right)^3 = \dfrac{d}{dx}\left(x^3 + \dfrac{1}{x^3} + 3x + \dfrac{3}{x}\right)$

$$= \frac{d}{dx}(x^3 + x^{-3} + 3x + 3 \cdot x^{-1}) = 3x^2 - 3x^{-4} + 3 - 3 \cdot x^{-2}$$

$$= 3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}$$

Hence, the required answer is $3x^2 - \dfrac{3}{x^4} + 3 - \dfrac{3}{x^2}$.

Q31. $(3x + 5)(1 + \tan x)$

Sol. $\dfrac{d}{dx}(3x + 5)(1 + \tan x)$

$$= (3x+5)\frac{d}{dx}(1+\tan x) + (1+\tan x)\frac{d}{dx}(3x+5)$$

$$= (3x + 5)\ (\sec^2 x) + (1 + \tan x)\ (3)$$

$$= 3x \sec^2 x + 5 \sec^2 x + 3 + 3 \tan x \quad \text{[using product rule]}$$

Hence, the required answer is $3x \sec^2 x + 5 \sec^2 x + 3 \tan x + 3$

Q32. $(\sec x - 1)\ (\sec x + 1)$

Sol. $\frac{d}{dx}(\sec x - 1)(\sec x + 1)$

$$= (\sec x - 1)\cdot\frac{d}{dx}(\sec x + 1) + (\sec x + 1)\frac{d}{dx}(\sec x - 1)$$

[using product rule]

$$= (\sec x - 1)\ (\sec x \tan x) + (\sec x + 1)\ (\sec x \tan x)$$

$$= \sec x \tan x\ (\sec x - 1 + \sec x + 1)$$

$$= \sec x \tan x \cdot 2 \sec x = 2 \sec^2 x \cdot \tan x$$

Hence, the required answer is $2 \tan x \sec^2 x$.

Q33. $\frac{3x+4}{5x^2-7x+9}$

Sol. $\frac{d}{dx}\left(\frac{3x+4}{5x^2-7x+9}\right)$

$$= \frac{(5x^2-7x+9)\frac{d}{dx}(3x+4) - (3x+4)\cdot\frac{d}{dx}(5x^2-7x+9)}{(5x^2-7x+9)^2}$$

[Using quotient rule]

$$= \frac{(5x^2-7x+9)(3) - (3x+4)(10x-7)}{(5x^2-7x+9)^2}$$

$$= \frac{15x^2 - 21x + 27 - 30x^2 + 21x - 40x + 28}{(5x^2-7x+9)^2}$$

$$= \frac{-15x^2 - 40x + 55}{(5x^2-7x+9)^2} = \frac{55 - 40x - 15x^2}{(5x^2-7x+9)^2}$$

Hence, the required answer is $\frac{55 - 40x - 15x^2}{(5x^2-7x+9)^2}$

Q34. $\frac{x^5 - \cos x}{\sin x}$

Sol. $\frac{d}{dx}\left(\frac{x^5 - \cos x}{\sin x}\right) = \frac{\sin x \cdot \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\cdot\frac{d}{dx}(\sin x)}{\sin^2 x}$

[Using quotient rule]

$$= \frac{\sin x\,(5x^4 + \sin x) - (x^5 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{5x^4 \cdot \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + (\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$$

Hence, the required answer is $\frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$.

Q35. $\frac{x^2 \cos \frac{\pi}{4}}{\sin x}$

Sol. $\frac{d}{dx}\left(\frac{x^2 \cos \frac{\pi}{4}}{\sin x}\right) = \cos \frac{\pi}{4} \cdot \frac{d}{dx}\left(\frac{x^2}{\sin x}\right)$

$$= \frac{\frac{1}{\sqrt{2}}\left[\sin x \cdot \frac{d}{dx}(x^2) - x^2 \cdot \frac{d}{dx}(\sin x)\right]}{\sin^2 x} \quad \text{[Using quotient rule]}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right] = \frac{1}{\sqrt{2}}\left[\frac{2x}{\sin x} - \frac{x^2 \cos x}{\sin^2 x}\right]$$

$$= \frac{1}{\sqrt{2}}[2x \operatorname{cosec} x - x^2 \cot x \operatorname{cosec} x]$$

$$= \frac{x}{\sqrt{2}} \cos ec\, x\,[2 - x \cot x]$$

Hence, the required answer is $\frac{x}{\sqrt{2}} \operatorname{cosec} x\,[2 - x \cot x]$.

Q36. $(ax^2 + \cot x)\,(p + q \cos x)$

Sol. $\frac{d}{dx}(ax^2 + \cot x)\,(p + q \cos x)$

$$= (ax^2 + \cot x)\frac{d}{dx}(p + q \cos x) + (p + q \cos x)\frac{d}{dx}(ax^2 + \cot x)$$

[Using Product Rule]

$$= (ax^2 + \cot x)\,(-q \sin x) + (p + q \cos x)\,(2ax - \operatorname{cosec}^2 x)$$

Hence, the required answer is

$$= (ax^2 + \cot x)\,(-q \sin x) + (p + q \cos x)\,(2ax - \operatorname{cosec}^2 x)$$

Q37. $\frac{a + b \sin x}{c + d \cos x}$

Sol. $\frac{d}{dx}\left(\frac{a + b \sin x}{c + d \cos x}\right)$

$$= \frac{(c + d\cos x)\cdot\frac{d}{dx}(a + b\sin x) - (a + b\sin x)\frac{d}{dx}(c + d\cos x)}{(c + d\cos x)^2}$$

[Using quotient rule]

$$= \frac{(c + d\cos x)(b\cos x) - (a + b\sin x)(-d\sin x)}{(c + d\cos x)^2}$$

$$= \frac{cb\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c + d\cos x)^2}$$

$$= \frac{cb\cos x + ad\sin x + bd(\cos^2 x + \sin^2 x)}{(c + d\cos x)^2}$$

$$= \frac{cb\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

Q38. $(\sin x + \cos x)^2$

Sol. $\frac{d}{dx}(\sin x + \cos x)^2 = \frac{d}{dx}(\sin x + \cos x)(\sin x + \cos x)$

$$= (\sin x + \cos x)\frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}(\sin x + \cos x)$$

$$= 2(\sin x + \cos x)\frac{d}{dx}(\sin x + \cos x)$$

$$= 2(\sin x + \cos x)(\cos x - \sin x) = 2(\cos^2 x - \sin^2 x) = 2\cos 2x$$

Hence, the required answer is $2\cos 2x$.

Q39. $(2x - 7)^2 (3x + 5)^3$

Sol. $\frac{d}{dx}(2x - 7)^2 (3x + 5)^3$

$$= (2x - 7)^2 \cdot \frac{d}{dx}(3x + 5)^3 + (3x + 5)^3 \cdot \frac{d}{dx}(2x - 7)^2$$

[Using product Rule]

$$= (2x - 7)^2 \cdot 3(3x + 5)^2 \cdot 3 + (3x + 5)^3 \cdot 2(2x - 7) . 2$$

$$= 9(2x - 7)^2 (3x + 5)^2 + 4(3x + 5)^3 (2x - 7)$$

$$= (2x - 7)(3x + 5)^2 [9(2x - 7) + 4(3x + 5)]$$

$$= (2x - 7)(3x + 5)^2 (18x - 63 + 12x + 20)$$

$$= (2x - 7)(3x + 5)^2 (30x - 43)$$

Hence, the required answer is $(2x - 7)(30x - 43)(3x + 5)^2$

Q40. $x^2 \sin x + \cos 2x$

Sol. $\frac{d}{dx}(x^2 \sin x + \cos 2x) = \frac{d}{dx}(x^2 \sin x) + \frac{d}{dx}(\cos 2x)$

$$= (x^2 \cos x + \sin x \cdot 2x) + (-2 \sin 2x)$$
$$= x^2 \cos x + 2x \sin x - 2 \sin 2x$$

Hence, the required answer is $x^2 \cos x + 2x \sin x - 2 \sin 2x$.

Q41. $\sin^3 x \cos^3 x$

Sol. $$\frac{d}{dx}(\sin^3 x \cos^3 x) = \sin^3 x \cdot \frac{d}{dx} \cos^3 x + \cos^3 x \cdot \frac{d}{dx}(\sin^3 x)$$

[Using Product Rule]

$$= \sin^3 x \cdot 3 \cos^2 x (-\sin x) + \cos^3 x \cdot 3 \sin^2 x \cdot \cos x$$
$$= -3 \sin^4 x \cos^2 x + 3 \cos^4 x \sin^2 x$$
$$= 3 \sin^2 x \cos^2 x (-\sin^2 x + \cos^2 x)$$
$$= 3 \sin^2 x \cos^2 x \cdot \cos 2x$$
$$= \frac{3}{4} \cdot 4 \sin^2 x \cos^2 x \cdot \cos 2x = \frac{3}{4}(2 \sin x \cos x)^2 \cos 2x$$
$$= \frac{3}{4} \sin^2 2x \cdot \cos 2x$$

Hence, the required answer is $\frac{3}{4} \sin^2 2x \cos 2x$

Q42. $\dfrac{1}{ax^2 + bx + c}$

Sol. $$\frac{d}{dx}\left(\frac{1}{ax^2 + bx + c}\right) = \frac{(ax^2 + bx + c)\frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2}$$

[Using quotient rule]

$$= \frac{(ax^2 + bx + c) \times 0 - (2ax + b)}{(ax^2 + bx + c)^2} = \frac{-(2ax + b)}{(ax^2 + bx + c)^2}$$

Hence, the required answer is $\dfrac{-(2ax + b)}{(ax^2 + bx + c)^2}$

LONG ANSWER TYPE QUESTIONS

Differentiate each of the functions with respect to 'x' in Exercise 43 to 46 using first principle method.

Q43. $\cos (x^2 + 1)$

Sol. Let $f(x) = \cos(x^2 + 1)$ ...(*i*)

$\Rightarrow$ $f(x + \Delta x) = \cos [(x + \Delta x)^2 + 1]$ ...(*ii*)

Subtracting eq. (*i*) from eq. (*ii*) we get

$$f(x + \Delta x) - f(x) = \cos [(x + \Delta x)^2 + 1] - \cos(x^2 + 1)$$

Dividing both sides by Δx we get

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\cos [(x + \Delta x)^2 + 1] - \cos(x^2 + 1)}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos[(x+\Delta x)^2 + 1] - \cos(x^2+1)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cos[(x+\Delta x)^2 + 1] - \cos(x^2+1)}{\Delta x}$$

[By definitions of differentiations]

$$= \lim_{\Delta x \to 0} \frac{-2\sin\left[\frac{(x+\Delta x)^2 + 1 + x^2 + 1}{2}\right] . \sin\left[\frac{(x+\Delta x)^2 + 1 - x^2 - 1}{2}\right]}{\Delta x}$$

$$\left[\because \quad \cos C - \cos D = -2\sin\frac{C+D}{2} \cdot \sin\frac{C-D}{2}\right]$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\left[\frac{x^2 + \Delta x^2 + 2x \cdot \Delta x + x^2 + 2}{2}\right] \cdot \sin\left[\frac{x^2 + \Delta x^2 + 2x\,\Delta x - x^2}{2}\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\left[x^2 + \frac{\Delta x^2}{2} + x\,\Delta x + 1\right]\sin\left[\Delta x\frac{(\Delta x + 2x)}{2}\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\left[x^2 + \frac{\Delta x^2}{2} + x\,\Delta x + 1\right] \cdot \sin\left[\Delta x\frac{(\Delta x + 2x)}{2}\right]}{\Delta x\left[\frac{\Delta x + 2x}{2}\right]} \times \left(\frac{\Delta x + 2x}{2}\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \therefore \Delta x\left[\frac{\Delta x + 2x}{2}\right] \to 0}} -2\sin\left[x^2 + \frac{\Delta x^2}{2} + x\Delta x + 1\right] \cdot \frac{\sin\left[\Delta x\frac{(\Delta x + 2x)}{2}\right]}{\Delta x\left[\frac{\Delta x + 2x}{2}\right]} \times \left[\frac{\Delta x + 2x}{2}\right]$$

Taking limit, we have

$= -2\sin(x^2+1) \cdot 1 \cdot (x) = -2x\sin(x^2+1)$ $\left[\because \quad \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

Hence, the required answer is $-2x\sin(x^2+1)$.

Q44. $\dfrac{ax+b}{cx+d}$

Sol. Let $\quad f(x) = \dfrac{ax+b}{cx+d} \quad \ldots(i)$

$\Rightarrow \quad f(x+\Delta x) = \dfrac{a(x+\Delta x)+b}{c(x+\Delta x)+d} \quad \ldots(ii)$

Subtracting eq. (*i*) from eq. (*ii*) we get

$$f(x+\Delta x) - f(x) = \frac{a(x+\Delta x)+b}{c(x+\Delta x)+d} - \frac{ax+b}{cx+d}$$

Dividing both sides by Δx and take the limit, we get

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{a(x+\Delta x)+b}{c(x+\Delta x)+d} - \dfrac{ax+b}{cx+d}}{\Delta x}$$

$$\Rightarrow \quad f'(x) = \lim_{\Delta x \to 0} \frac{(ax+a\Delta x+b)(cx+d)-(ax+b)(cx+c\Delta x+d)}{[c(x+\Delta x)+d](cx+d)\cdot \Delta x}$$

[Using definition of differentiation]

$$= \lim_{\Delta x \to 0} \frac{acx^2 + ac\,\Delta x \cdot x + bcx + adx + ad\Delta x + bd - acx^2 - ac\Delta x \cdot x - adx - bcx - bc \cdot \Delta x - bd}{(cx+c\Delta x+d)(cx+d)\cdot \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(ad-bc)\,\Delta x}{(cx+c\Delta x+d)(cx+d)\cdot \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(ad-bc)}{(cx+c\cdot\Delta x+d)(cx+d)}$$

Taking limit, we have

$$= \frac{(ad-bc)}{(cx+d)(cx+d)} = \frac{ad-bc}{(cx+d)^2}$$

Hence, the required answer is $\dfrac{ad-bc}{(cx+d)^2}$.

Q45. $x^{2/3}$

Sol. Let $\quad f(x) = x^{2/3} \quad \ldots(i)$

$f(x+\Delta x) = (x+\Delta x)^{2/3} \quad \ldots(ii)$

Subtracting eq. (*i*) from (*ii*) we get

$$f(x+\Delta x) - f(x) = (x+\Delta x)^{2/3} - x^{2/3}$$

Dividing both sides by Δx and take the limit.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{2/3} - x^{2/3}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^{2/3}\left[1+\frac{\Delta x}{x}\right]^{2/3} - x^{2/3}}{\Delta x}$$

[By definition of differentiation]

$$= \lim_{\Delta x \to 0} \frac{x^{2/3}\left[\left(1+\frac{\Delta x}{x}\right)^{2/3} - 1\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^{2/3}\left[\left(1+\frac{2}{3}\cdot\frac{\Delta x}{x} + \cdots\right) - 1\right]}{\Delta x}$$

[Expanding by Binomial theorem and rejecting the higher powers of Δx as $\Delta x \to 0$]

$$= \lim_{\Delta x \to 0} \frac{x^{2/3}\cdot\frac{2}{3}\cdot\frac{\Delta x}{x}}{\Delta x} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

Hence, the required answer is $\frac{2}{3}x^{-1/3}$.

Q46. $x \cos x$

Sol. Let $y = x \cos x$ $\quad (i)$

$y + \Delta y = (x + \Delta x)\cos(x + \Delta x)$ $\quad (ii)$

Subtracting eq. (*i*) from eq. (*ii*) we get

$y + \Delta y - y = (x + \Delta x)\cos(x + \Delta x) - x\cos x$

$\Rightarrow \quad \Delta y = x\cos(x + \Delta x) + \Delta x\cos(x + \Delta x) - x\cos x$

Dividing both sides by Δx and take the limits,

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{x\cos(x+\Delta x) - x\cos x + \Delta x\cos(x+\Delta x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{x[\cos(x+\Delta x) - \cos x]}{\Delta x} + \lim_{\Delta x \to 0}\frac{\Delta x\cos(x+\Delta x)}{\Delta x}$$

$\left[\text{By definition} \quad \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right]$

$$= \lim_{\Delta x \to 0}\frac{x\left[-2\sin\frac{(x+\Delta x+x)}{2}\cdot\sin\frac{(x+\Delta x - x)}{2}\right]}{\Delta x} + \lim_{\Delta x \to 0}\cos(x+\Delta x)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \therefore \frac{\Delta x}{2} \to 0}}\frac{x\left[-2\sin\left(x+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}\right]}{2\times\frac{\Delta x}{2}} + \lim_{\Delta x \to 0}\cos(x+\Delta x)$$

$\therefore \quad \frac{\Delta x}{2} \to 0$ Taking the limits, we have

$$= x[-\sin x] + \cos x \qquad \left[\because \quad \lim_{\frac{\Delta x}{2} \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1\right]$$

$$= -x \sin x + \cos x$$

Hence, the required answer is $-x \sin x + \cos x$

Evaluate each of the following limits in Exercise 47 to 53.

Q47. $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y}$

Sol. $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y}$

$$= \lim_{y \to 0} \frac{x \sec(x+y) + y \sec(x+y) - x \sec x}{y}$$

$$= \lim_{y \to 0} \frac{[x \sec(x+y) - x \sec x]}{y} + \lim_{y \to 0} \frac{y \sec(x+y)}{y}$$

$$= \lim_{y \to 0} \frac{x[\sec(x+y) - \sec x]}{y} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \frac{x\left[\frac{1}{\cos(x+y)} - \frac{1}{\cos x}\right]}{y} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} x\left[\frac{\cos x - \cos(x+y)}{y \cdot \cos(x+y) \cdot \cos x}\right] + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \frac{x\left[-2\sin\left(\frac{x+x+y}{2}\right) \cdot \sin\left(\frac{x-x-y}{2}\right)\right]}{y \cos(x+y) \cdot \cos x} + \lim_{y \to 0} \sec(x+y)$$

$$= \frac{x\left[-2\sin\left(x+\frac{y}{2}\right) \cdot \sin\left(-\frac{y}{2}\right)\right]}{\cos(x+y) \cdot \cos x \cdot y} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{\substack{y \to 0 \\ \therefore \frac{y}{2} \to 0}} x\left[\frac{\left[2\sin\left(x+\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)\right]}{\cos(x+y) \cdot \cos x \cdot \left(\frac{y}{2}\right) \cdot 2}\right] + \lim_{y \to 0} \sec(x+y)$$

$\therefore$ Taking the limits we have

$$= x\left[\sin x \cdot \frac{1}{\cos x \cdot \cos x}\right] + \sec x$$

$= x \sec x \tan x + \sec x = \sec x\ (x \tan x + 1)$

Hence, the required answer is $\sec x\ (x \tan x + 1)$

Q48. $\displaystyle \lim_{x \to 0} \frac{(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha \cdot x)}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

Sol. Given, $\displaystyle \lim_{x \to 0} \frac{[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha \cdot x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

$$= \lim_{x \to 0} \frac{[2 \sin \alpha x \cdot \cos \beta x + \sin 2\alpha \cdot x] \cdot x}{2 \sin(\alpha+\beta)x \cdot \sin(\alpha-\beta)\, x}$$

$$\left[\begin{array}{l} \because \quad \sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cdot \cos \dfrac{C-D}{2} \\ \quad\quad \cos C - \cos D = -2 \sin \dfrac{C+D}{2} \cdot \sin \dfrac{C-D}{2} \end{array}\right]$$

$$= \lim_{x \to 0} \frac{[2 \sin \alpha x \cdot \cos \beta x + 2 \sin \alpha x \cdot \cos \alpha x] \cdot x}{2 \sin(\alpha+\beta)x \cdot \sin(\alpha-\beta)x}$$

$$= \lim_{x \to 0} \frac{2 \sin \alpha x\, (\cos \beta x + \cos \alpha x) \cdot x}{2 \sin(\alpha+\beta)x \cdot \sin(\alpha-\beta)x}$$

$$= \lim_{x \to 0} \frac{\sin \alpha x \left[2 \cos\left(\dfrac{\alpha+\beta}{2}\right) x \cdot \cos\left(\dfrac{\alpha-\beta}{2}\right) x\right] \cdot x}{\sin(\alpha+\beta)x \cdot \sin(\alpha-\beta)x}$$

$$= \lim_{x \to 0} \frac{\sin \alpha x \left[2 \cos\left(\dfrac{\alpha+\beta}{2}\right) x \cdot \cos\left(\dfrac{\alpha-\beta}{2}\right) x\right] \cdot x}{2 \sin\left(\dfrac{\alpha+\beta}{2}\right) x \cdot \cos\left(\dfrac{\alpha+\beta}{2}\right) x \cdot 2 \sin\left(\dfrac{\alpha-\beta}{2}\right) x \cdot \cos\left(\dfrac{\alpha-\beta}{2}\right) x}$$

$$\left[\begin{array}{l} \because \quad \cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cdot \cos \dfrac{C-D}{2} \\ \text{and } \sin 2x = 2 \sin x \cos x \end{array}\right]$$

$$= \lim_{x \to 0} \frac{\sin \alpha x \cdot x}{2 \sin\left(\dfrac{\alpha+\beta}{2}\right) x \sin\left(\dfrac{\alpha-\beta}{2}\right) \cdot x}$$

$$= \lim_{x\to 0} \frac{1}{2} \frac{\frac{\sin \alpha x}{\alpha x}\cdot(\alpha x)\cdot x}{\left[\frac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)\cdot x}\times\left(\frac{\alpha+\beta}{2}\right)\cdot x\right]\left[\frac{\sin\left(\frac{\alpha-\beta}{2}\right)\cdot x}{\left(\frac{\alpha-\beta}{2}\right)\cdot x}\times\frac{(\alpha-\beta)}{2}\cdot x\right]}$$

$$= \frac{1}{2}\cdot\frac{\alpha x^2}{\left(\frac{\alpha+\beta}{2}\right)x\cdot\left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{1}{2}\left[\frac{\alpha}{\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)}\right]$$

$$= \frac{1}{2}\cdot\frac{4\alpha}{\alpha^2-\beta^2} = \frac{2\alpha}{\alpha^2-\beta^2}$$

Hence, the required answer is $\frac{2\alpha}{\alpha^2-\beta^2}$

Q49. $\lim_{x\to\frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

Sol. Given, $\lim_{x\to\frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\tan x\,(\tan^2 x - 1)}{\cos\left(x+\frac{\pi}{4}\right)} = \lim_{x\to\frac{\pi}{4}} \tan x \cdot \lim_{x\to\frac{\pi}{4}} \left[\frac{-(1-\tan^2 x)}{\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= -1\times \lim_{x\to\frac{\pi}{4}} \frac{(1-\tan x)\,(1+\tan x)}{\cos\left(x+\frac{\pi}{4}\right)}$$

$$= \lim_{x\to\frac{\pi}{4}} -(1+\tan x)\cdot \lim_{x\to\frac{\pi}{4}}\left(\frac{1-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}\right)$$

$$= -(1+1)\cdot \lim_{x\to\frac{\pi}{4}} \frac{(\cos x - \sin x)}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}$$

$$= -2 \times \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -2\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\left[\cos \frac{\pi}{4} \cdot \cos x - \sin \frac{\pi}{4} \sin x\right]}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-2\sqrt{2} \cdot \cos\left(x + \frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = \frac{-2\sqrt{2}}{\cos \frac{\pi}{4}} \quad \text{(Taking limit)}$$

$$= \frac{-2\sqrt{2}}{\frac{1}{\sqrt{2}}} = -2 \times 2 = -4$$

Hence, the required answer is -4.

Q50. $$\lim_{x \to \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2}\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)}$$

Sol. Given, $$\lim_{x \to \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2}\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)}$$

$$= \lim_{x \to \pi} \frac{\cos^2 \frac{x}{4} + \sin^2 \frac{x}{4} - 2 \sin \frac{x}{4} \cdot \cos \frac{x}{4}}{\left(\cos^2 \frac{x}{4} - \sin^2 \frac{x}{4}\right)\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)}$$

$[\because \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta]$

$$= \lim_{x \to \pi} \frac{\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)^2}{\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right) \cdot \left(\cos \frac{x}{4} + \sin \frac{x}{4}\right) \cdot \left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)}$$

$$= \lim_{x \to \pi} \frac{1}{\left(\cos \frac{x}{4} + \sin \frac{x}{4}\right)}$$

Taking limits we have

$$= \frac{1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{1}{\frac{2}{\sqrt{2}}} = \frac{1}{\sqrt{2}}$$

Hence, the required answer is $\frac{1}{\sqrt{2}}$.

Q51. Show that $\displaystyle\lim_{x\to 4} \frac{|x-4|}{x-4}$ does not exist.

Sol. Given $\displaystyle\lim_{x\to 4} \frac{|x-4|}{x-4}$

$$\text{LHL} = \lim_{x\to 4^-} \frac{-(x-4)}{x-4} = -1 \qquad [\because\ |x-4| = -(x-4) \text{ if } x < 4]$$

$$\text{RHL} = \lim_{x\to 4^+} \frac{x-4}{x-4} = 1 \qquad [\because\ |x-4| = (x-4) \text{ if } x > 4]$$

Since LHL $\neq$ RHL

Hence, the limit does not exist.

Q52. If $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{when } x \neq \dfrac{\pi}{2} \\ 3, & x = \dfrac{\pi}{2} \end{cases}$ and if $f(x) = f\left(\dfrac{\pi}{2}\right)$

Find the value of k.

Sol. Given, $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & x \neq \dfrac{\pi}{2} \\ 3, & x = \dfrac{\pi}{2} \end{cases}$

$$\text{LHL } f(x) = \lim_{x\to \frac{\pi^-}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h\to 0} \frac{k\sin h}{\pi - \pi + 2h} = \lim_{h\to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2}\cdot 1 = \frac{k}{2} \qquad \left[\because\ \lim_{h\to 0}\frac{\sin h}{h} = 1\right]$$

$$\text{RHL } f(x) = \lim_{x\to \frac{\pi^+}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h\to 0} \frac{-k \sin h}{\pi - \pi - 2h}$$

$$= \lim_{h\to 0} \frac{-k \sin h}{-2h} = \frac{k}{2} \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

we are given that $\lim_{x\to\frac{\pi}{2}} f(x) = 3$

So, $\frac{k}{2} = 3 \quad \Rightarrow \quad k = 6$

Hence, the required answer is 6.

Q53. If $f(x) = \begin{cases} x+2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$ then find c when $\lim_{x\to -1} f(x)$ exists.

Sol. Given, $f(x) = \begin{cases} x+2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$

$$\text{LHL} = \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x+2)$$

$$= \lim_{h\to 0} (-1 - h + 2) = \lim_{h\to 0} (1-h) = 1$$

$$\text{RHL} = \lim_{x\to -1^+} cx^2 = \lim_{h\to 0} c(-1+h)^2 = c$$

Since the limits exist.

$\therefore$ LHL = RHL

$\therefore$ $c = 1$

Hence, the required answer = 1

OBJECTIVE TYPE QUESTIONS

Q54. $\lim_{x\to\pi} \frac{\sin x}{x - \pi}$ is equal to

(*a*) 1 (*b*) 2 (*c*) −1 (*d*) −2

Sol. Given, $\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{x\to\pi} \frac{\sin(\pi - x)}{-(\pi - x)}$

$$= -1 \quad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1 \text{ and } \pi - x \to 0 \quad \Rightarrow x \to \pi\right]$$

Hence, the correct option is (*c*).

Q55. $\lim_{x\to 0} \frac{x^2 \cos x}{1 - \cos x}$ is equal to

(*a*) 2 (*b*) 3/2 (*c*) −3/2 (*d*) 1

Sol. Given $\lim_{x\to 0} \frac{x^2 \cos x}{1 - \cos x} = \lim_{x\to 0} \frac{x^2 \cos x}{2\sin^2 \frac{x}{2}} \quad \left[\because 1 - \cos x = 2\sin^2 \frac{x}{2}\right]$

$$= \lim_{x \to 0} \frac{\frac{x^2}{4} \times 4 \cos x}{2 \sin^2 \frac{x}{2}} = \lim_{x \to 0 \Rightarrow \frac{x}{2} \to 0} \frac{\left(\frac{x}{2}\right)^2 \cdot 2 \cos x}{\sin^2 \frac{x}{2}}$$

$$= \lim_{\frac{x}{2} \to 0} \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 \cdot 2 \cos x$$

$$= 2 \cos 0 = 2 \times 1 = 2 \qquad \left[\because \lim_{x \to 0} \frac{x}{\sin x} = 1\right]$$

Hence, the correct option is (*a*).

Q56. $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$ is equal to

(*a*) n (*b*) 1 (*c*) $-n$ (*d*) 0

Sol. Given $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{x \to 0} \frac{(1+x)^n - (1)^n}{(1+x) - (1)}$

$$= \lim_{1+x \to 1} \frac{(1+x)^n - (1)^n}{(1+x) - (1)} = n(1)^{n-1} = n \left[\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Hence, the correct option is (*a*).

Q57. $\lim_{x \to 1} \frac{x^m - 1}{x^n - 1}$ is equal to

(*a*) 1 (*b*) m/n (*c*) $-m/n$ (*d*) m^2/n^2

Sol. Given $\lim_{x \to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \to 1} \frac{\frac{x^m - (1)^m}{x - 1}}{\frac{x^n - (1)^n}{x - 1}}$

$$= \frac{m(1)^{m-1}}{n(1)^{n-1}} = \frac{m}{n} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Hence, the correct option is (*b*).

Q58. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is equal to

(*a*) 4/9 (*b*) 1/2 (*c*) $-1/2$ (*d*) -1

Sol. Given $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \to 0} \frac{2 \sin^2 2\theta}{2 \sin^2 3\theta}$

$$\left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}\right]$$

$$= \lim_{\theta \to 0} \frac{\sin^2 2\theta}{\sin^2 3\theta} = \lim_{\theta \to 0} \left[\frac{\sin 2\theta}{\sin 3\theta}\right]^2$$

$$= \lim_{\substack{\theta \to 0 \\ 2\theta \to 0 \\ 3\theta \to 0}} \left[\frac{\frac{\sin 2\theta}{2\theta} \times 2\theta}{\frac{\sin 3\theta}{3\theta} \times 3\theta}\right]^2 = \left[\frac{2\theta}{3\theta}\right]^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Hence, the correct option is (*a*).

Q59. $\lim_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$ is equal to

(*a*) $-\frac{1}{2}$ (*b*) 1 (*c*) $\frac{1}{2}$ (*d*) -1

Sol. Given $\lim_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x} = \lim_{x \to 0} \dfrac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \frac{2 \sin^2 \frac{x}{2}}{x \cdot 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$[\because \quad \sin 2x = 2 \sin x \cos x]$

$$= \lim_{x \to 0} \frac{\sin \frac{x}{2}}{x \cos \frac{x}{2}} = \lim_{x \to 0} \frac{\tan \frac{x}{2}}{x} = \lim_{x \to 0} \frac{\tan \frac{x}{2}}{2 \times \frac{x}{2}}$$

$$= \frac{1}{2} \times 1 = \frac{1}{2} \qquad \left[\because \quad \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

Hence, the correct option is (*c*).

Q60. $\lim_{x \to 0} \dfrac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is

(*a*) 2 (*b*) 0 (*c*) 1 (*d*) -1

Sol. Given $\lim_{x \to 0} \dfrac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$

$$= \lim_{x \to 0} \frac{\sin x \left[\sqrt{x+1} + \sqrt{1-x}\right]}{(\sqrt{x+1} - \sqrt{1-x})(\sqrt{x+1} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{\sin x \left[\sqrt{x+1} + \sqrt{1-x}\right]}{x + 1 - 1 + x}$$

$$= \lim_{x\to 0} \frac{\sin x\left[\sqrt{x+1}+\sqrt{1-x}\right]}{2x} = \frac{1}{2}\cdot\lim_{x\to 0}\frac{\sin x}{x}[\sqrt{x+1}+\sqrt{1-x}]$$

Taking limit, we get

$$= \frac{1}{2}\times 1\times\left[\sqrt{0+1}+\sqrt{1-0}\right] = \frac{1}{2}\times 1\times 2 = 1$$

Hence, the correct option is (*c*).

Q61. $\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is equal to

(*a*) 3 (*b*) 1 (*c*) 0 (*d*) 2

Sol. Given, $\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1} = \lim_{x\to\frac{\pi}{4}} \frac{1+\tan^2 x - 2}{\tan x - 1}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\tan^2 x - 1}{\tan x - 1} = \lim_{x\to\frac{\pi}{4}} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)}$$

$$= \lim_{x\to\frac{\pi}{4}} (\tan x + 1) = \tan\frac{\pi}{4} + 1 = 1 + 1 = 2$$

Hence, the correct option is (*d*).

Q62. $\lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$

(*a*) $\frac{1}{10}$ (*b*) $-\frac{1}{10}$

(*c*) 1 (*d*) None of these

Sol. Given $\lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+3x-2x-3}$

$$= \lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{x(2x+3)-1(2x+3)} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{(x-1)(2x+3)}$$

$$= \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)} = \lim_{x\to 1} \frac{(x-1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)}$$

$$= \lim_{x\to 1} \frac{2x-3}{(\sqrt{x}+1)(2x+3)}$$

Taking limit we have

$$= \frac{2(1)-3}{(\sqrt{1}+1)(2\times 1+3)} = \frac{-1}{2\times 5} = \frac{-1}{10}$$

Hence, the correct option is (*b*).

Q63. If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$ where $[\cdot]$ denotes the greatest integer function. Then $\lim\limits_{x \to 0} f(x)$

(*a*) 1 (*b*) 0

(*c*) -1 (*d*) None of these

Sol. Given, $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} \frac{\sin[x]}{[x]} = \lim_{h \to 0} \frac{\sin[0-h]}{[0-h]} = \lim_{h \to 0} \frac{-\sin[-h]}{[-h]} = -1$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\sin[x]}{[x]} = \lim_{h \to 0} \frac{\sin[0+h]}{[0+h]} = \lim_{h \to 0} \frac{\sin[h]}{[h]} = 1$$

$$\text{LHL} \neq \text{RHL}$$

So, the limit does not exist.

Hence, the correct option is (*d*).

Q64. $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$ is equal to

(*a*) 1 (*b*) -1

(*c*) Does not exist (*d*) None of these.

Sol. Given $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$

$$\text{LHL} = \lim_{x \to 0^-} \frac{-\sin x}{x} = -1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

LHL $\neq$ RHL, so the limit does not exist.

Hence, the correct option is (*c*).

Q65. If $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$, then the quadratic equation whose roots are $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ is

(*a*) $x^2 - 6x + 9 = 0$ (*b*) $x^2 - 7x + 8 = 0$

(*c*) $x^2 - 14x + 49 = 0$ (*d*) $x^2 - 10x + 21 = 0$

Sol. Given $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$

$$\therefore \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 1)$$

$$\lim_{h\to 0} [(2-h)^2 - 1] = \lim_{h\to 0} (4 + h^2 - 4h - 1)$$
$$= \lim_{h\to 0} (h^2 - 4h + 3) = 3$$

and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x+3) = \lim_{h\to 0} [2(2+h)+3] = 7$

Therefore, the quadratic equation whose roots are 3 and 7 is $x^2 - (3+7)x + 3\times 7 = 0$ i.e., $x^2 - 10x + 21 = 0$. Hence, the correct option is (*d*).

Q66. $\lim_{x\to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ is equal to

(*a*) 2 (*b*) $\dfrac{1}{2}$ (*c*) $-\dfrac{1}{2}$ (*d*) $\dfrac{1}{4}$

Sol. Given $\lim_{x\to 0} \dfrac{\tan 2x - x}{3x - \sin x} = \lim_{x\to 0} \dfrac{x\left[\dfrac{\tan 2x}{x} - 1\right]}{x\left[3 - \dfrac{\sin x}{x}\right]}$

$$\lim_{\substack{x\to 0\\ \therefore 2x\to 0}} \frac{\dfrac{\tan 2x}{2x}\times 2 - 1}{3 - \dfrac{\sin x}{x}} = \frac{1.2 - 1}{3-1} = \frac{2-1}{2} = \frac{1}{2}$$

$\therefore \quad 2x \to 0$

Hence, the correct option is (*b*).

Q67. If $f(x) = x - [x]$, $\in$ R then $f'\left(\dfrac{1}{2}\right)$ is equal to

(*a*) $\dfrac{3}{2}$ (*b*) 1 (*c*) 0 (*d*) -1

Sol. Given $f(x) = x - [x]$

we have to first check for differentiability of $f(x)$ at $x = 1/2$

$$\therefore\ Lf'\left(\frac{1}{2}\right) = \text{LHD} = \lim_{h\to 0} \frac{f\left[\frac{1}{2} - h\right] - f\left[\frac{1}{2}\right]}{-h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{2} - h\right) - \left[\frac{1}{2} - h\right] - \frac{1}{2} + \left[\frac{1}{2}\right]}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2} - h - 0 - \frac{1}{2} + 0}{-h} = \frac{-h}{-h} = 1$$

$$Rf'\left(\frac{1}{2}\right) = \text{RHD} = \lim_{h\to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{2}+h\right)-\left[\frac{1}{2}+h\right]-\frac{1}{2}+\left[\frac{1}{2}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}+h-1-\frac{1}{2}+1}{h} = \frac{h}{h} = 1$$

Since LHD = RHD

$\therefore$ $f'\left(\frac{1}{2}\right) = 1$

Hence, the correct option is (*b*).

Q68. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(*a*) 1 (*b*) $\frac{1}{2}$ (*c*) $\frac{1}{\sqrt{2}}$ (*d*) 0

Sol. Given that $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{1}{2} - \frac{1}{2} = 0$$

Hence, the correct option is (*d*).

Q69. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is equal to

(*a*) $\frac{5}{4}$ (*b*) $\frac{4}{5}$ (*c*) 1 (*d*) 0

Sol. Given that $f(x) = \frac{x-4}{2\sqrt{x}}$

$$\therefore \quad f'(x) = \frac{1}{2}\left[\frac{\sqrt{x}\cdot 1-(x-4)\cdot\frac{1}{2\sqrt{x}}}{x}\right]$$

$$= \frac{1}{2}\left[\frac{2x-x+4}{2\sqrt{x}\cdot x}\right] = \frac{1}{2}\left[\frac{x+4}{2(x)^{3/2}}\right]$$

$$\therefore \quad f'(x) \text{ at } x = 1 = \frac{1}{2}\left[\frac{1+4}{2\times 1}\right] = \frac{5}{4}$$

Hence, the correct option is (*a*).

Q70. If $y = \dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$ then $\frac{dy}{dx}$ is equal to (*a*) $\dfrac{-4x}{(x^2-1)^2}$ (*b*) $\dfrac{-4x}{x^2-1}$

(*c*) $\dfrac{1-x^2}{4x}$ (*d*) $\dfrac{4x}{x^2-1}$

Sol. Given $y = \dfrac{1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}} \Rightarrow y = \dfrac{x^2+1}{x^2-1}$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2-1)\cdot 2x - (x^2+1)\cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Hence, the correct option is (*a*).

Q71. If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(*a*) -2 (*b*) 0

(*c*) 1/2 (*d*) Does not exist

Sol. Given $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

$$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{-[\sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=0} = \frac{-2}{(\sin 0 - \cos 0)^2} = \frac{-2}{(-1)^2} = -2$$

Hence, the correct option is (*a*).

Q72. If $y = \dfrac{\sin(x+9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(*a*) cos 9 (*b*) sin 9 (*c*) 0 (*d*) 1

Sol. Given $$y = \frac{\sin(x+9)}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos(x+9) - \sin(x+9)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos x \cos(x+9) + \sin x \sin(x+9)}{\cos^2 x}$$

$$= \frac{\cos(x+9-x)}{\cos^2 x} = \frac{\cos 9}{\cos^2 x}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=0} = \frac{\cos 9}{\cos^2 0} = \frac{\cos 9}{(1)^2} = \cos 9$$

Hence, the correct option is (*a*).

Q73. If $f(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(*a*) $\frac{1}{100}$ (*b*) 100

(*c*) does not exist (*d*) 0

Sol. Given $$f(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{100}}{100}$$

$$f'(x) = 1 + \frac{2x}{2} + \cdots + \frac{100x^{99}}{100}$$

$$\therefore \quad f'(1) = 1 + 1 + 1 + \cdots + 1 \text{ (100 times)} = 100$$

Hence, the correct option is (*b*).

Q74. If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant, a, then $f'(a)$ is equal to

(*a*) 1 (*b*) 0

(*c*) does not exist (*d*) 1/2

Sol. Given $$f(x) = \frac{x^n - a^n}{x - a}$$

$$f'(x) = \frac{(x-a)(n \cdot x^{n-1}) - (x^n - a^n) \cdot 1}{(x-a)^2}$$

$$\therefore \quad f'(a) = \frac{(a-a)(n \cdot a^{n-1}) - (a^n - a^n)}{(a-a)^2}$$

So $$f'(a) = \frac{0}{0} = \text{does not exist}$$

Hence, the correct option is (c).

Q75. If $f(x) = x^{100} + x^{99} \cdots + x + 1$, then $f'(1)$ is equal to

(*a*) 5050 (*b*) 5049 (*c*) 5051 (*d*) 50051

Sol. Given, $f(x) = x^{100} + x^{99} + \cdots + x + 1$

$\therefore \quad f'(x) = 100x^{99} + 99.x^{98} + \cdots + 1$

So, $f'(1) = 100 + 99 + 98 + \ldots + 1$

$$= \frac{100}{2}[2 \times 100 + (100 - 1)(-1)]$$

$$= 50[200 - 99] = 50 \times 101 = 5050$$

Hence, the correct option is (*a*).

Q76. If $f(x) = 1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100}$, then $f'(1)$ is equal to

(*a*) 150 (*b*) -50 (*c*) -150 (*d*) 50

Sol. Given that $f(x) = 1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100}$

$$f'(x) = -1 + 2x - 3x^2 + \cdots - 99x^{98} + 100\,x^{99}$$

$$\therefore \quad f'(1) = -1 + 2 - 3 + \cdots - 99 + 100$$

$$= (-1 - 3 - 5 \cdots - 99) + (2 + 4 + 6 + \cdots + 100)$$

$$= \frac{50}{2}[2 \times -1 + (50 - 1)(-2)] + \frac{50}{2}[2 \times 2 + (50 - 1)2]$$

$$= 25[-2 - 98] + 25\,[4 + 98] = 25 \times -100 + 25 \times 102$$

$$= 25[-100 + 102] = 25 \times 2 = 50$$

Hence, the correct option is (*d*).

FILL IN THE BLANKS

Q77. If $f(x) = \dfrac{\tan x}{x - \pi}$, then $\lim\limits_{x \to \pi} f(x) = \cdots$

Sol. Given $f(x) = \lim\limits_{x \to \pi} \dfrac{-\tan(\pi - x)}{x - \pi} = \lim\limits_{\pi - x \to 0} \dfrac{-\tan(\pi - x)}{-(\pi - x)} = 1$

Hence, the value of the filler is 1.

Q78. $\lim\limits_{x \to 0}\left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$ then $m = \ldots$

Sol. Given $\lim\limits_{x \to 0}\left(\sin mx \cdot \cot \dfrac{x}{\sqrt{3}}\right) = 2$

$$= \lim_{\substack{x \to 0 \\ \therefore\, mx \to 0}} \frac{\sin mx}{mx} \times mx \lim_{x \to 0}\left(\cot \frac{x}{\sqrt{3}}\right) = 2$$

$$= 1 \times mx \times \lim_{x \to 0} \frac{1}{\tan \frac{x}{\sqrt{3}}} = 2$$

$$= \lim_{x \to 0} mx \times \frac{\frac{x}{\sqrt{3}}}{\frac{x}{\sqrt{3}} \cdot \tan \frac{x}{\sqrt{3}}} = 2$$

$$= \frac{mx}{\frac{x}{\sqrt{3}}}(1) = 2 \Rightarrow \sqrt{3}m = 2 \quad \Rightarrow \quad m = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Hence, the value of the filler is $\frac{2\sqrt{3}}{3}$.

Q79. If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, then $\frac{dy}{dx} = \cdots$

Sol. Given that $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\frac{dy}{dx} = 0 + \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = y$$

Hence the value of the filler is y.

Q80. $\lim_{x \to 3^+} \frac{x}{[x]}$

Sol. Given $\lim_{x \to 3^+} \frac{x}{[x]}$

$$= \lim_{\to} \frac{}{[3 \quad]} = \frac{}{3} = 1$$

Hence, the value of the filler is 1.